On Weakly $\pi$-Regular Rings

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Abstract:
In this paper, we continue, like several other authors, to study weakly $\pi$-regular rings. In particular, we investigate some characterizations and several basic properties of these rings and the relationship between them and simple rings, strongly $\pi$-regular rings, the maximality of prime ideals in ERT rings, exchange rings and Kasch rings, respectively.

Keywords: Weakly $\pi$-regular rings, strongly $\pi$-regular rings, reduced rings and ERT-rings.

1. Introduction

The relationship between various generalizations of Von Neumann regularity and the condition that every prime ideal is maximal have been investigated by many authors.\textsuperscript{[1,2,3,4]} The first clearly established equivalence between a generalization of Von Neumann regularity and the maximality of prime ideals seems to have been made by Storrer\textsuperscript{[5]} in the following result: If $R$ is a commutative ring with identity then $R$ is $\pi$-regular if and only if every prime ideal of $R$ is maximal. Storrer’s result was extended to P. I.-rings\textsuperscript{[2]} (Theorem 2.3) and bounded weakly right duo rings\textsuperscript{[3]} (Theorem 3), respectively. Recently, Birkenmeier et. al.\textsuperscript{[1]} showed that if $R$ is a 2-primal ring, then $R/P(R)$ is right weakly $\pi$-regular if and only if every prime ideal of $R$ is maximal. These results mainly explained the relation between the $\pi$-regularity and the maximality of prime ideals of rings.

The $\pi$-regularity of rings is extended to the weak $\pi$-regularity. In general, $\pi$-regular rings are weakly $\pi$-regular rings but the converse does not hold.

We investigate the connections between the results of previously mentioned papers and weak $\pi$-regularity in exchange rings, Kasch rings, locally finite rings and IFP rings, respectively. Consequently, our results in this paper have extended many of the results arrived at by other authors.\textsuperscript{[6,1,2,3]} Throughout this paper the letter $R$ denotes an associative ring with identity, and all prime ideals of $R$ are assumed to be proper. $P(R)$ and $J(R)$ denote the prime radical and the Jacobson radical of $R$, respectively. For any non-empty subset $X$ of a ring $R$, the right (left) annihilator of $X$ will be denoted by $r(X)$ ($l(X)$).

Recall that:

1- A ring $R$ is called (strongly) $\pi$-regular\textsuperscript{[7]} if for every $a \in R$, there exists a positive integer $n$, depending on $a$ such that $(a^n \in a^{n+1} R) \in a^n R a^n$.

Strongly $\pi$-regular is right-left symmetric.

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2- A ring $R$ is called reduced if it has no non-zero nilpotent elements.
3- A ring $R$ is called semi-prime if it has no non-zero nilpotent ideals.
4- A ring $R$ is called right (left) quasi-duo \cite{4} if every maximal right (left) ideal of $R$ is a two-sided ideal.
5- A ring $R$ is called ERT-ring \cite{3} if every essential right ideal of $R$ is a two-sided ideal.
6- A ring $R$ is called reversible \cite{8} if $ab = 0$ implies $ba = 0$ for $a, b \in R$.
7- A ring $R$ is said to be locally finite if every finite subset in it generates a finite semi-group multiplicatively.\cite{9}

2. Characterizations and Basic Properties

Following \cite{10}, a ring $R$ is said to be right (left) weakly $\pi$-regular if for every $a \in R$, there exists a positive integer $n$ such that $a^n \in a^n Ra^n R (a^n \in Ra^n Ra^n)$. $R$ is called weakly $\pi$-regular if it is both right and left weakly $\pi$-regular.

We now consider some new characterizations and several basic properties of weakly $\pi$-regular rings.

Theorem 2.1:
Let $R$ be a reduced ring. Then, $R$ is weakly $\pi$-regular if and only if for every $a \in R$, $r(a^n) \oplus Ra^n R = R$, for some positive integer $n$.

Proof:
Assume that $R = r(a^n) \oplus Ra^n R$, then $d + ba^n c = 1$, for some $b, c \in R$ and $d \in r(a^n)$. So, $a^n d + a^n ba^n c = a^n$. Thus, $a^n = a^n ba^n c$, whence $R$ is weakly $\pi$-regular.

Conversely, assume that $R$ is weakly $\pi$-regular ring. Then, for every $x \in R$, there exists $y \in Ra^n R$ and a positive integer $n$ such that $x^n = x^n y$. This implies that $x^n (1 - y) = 0$ and hence $(1 - y) \in r(x^n)$. Therefore, $r(x^n) + Rx^n R = R$. Let $z \in r(x^n) \cap Rx^n R$. Then, $x^n z = 0$ and hence $x^n z t = 0$, for some $t \in R$. This implies that $z t \in r(x^n) = r(x^n)$ since $R$ is reduced. So, $z Rx^n = (0)$ and $z Rx^n R = (0)$, then $z^2 = 0$. Since $R$ is reduced, then $b = 0$. Therefore, $r(x^n) \cap Rx^n R = (0)$. Hence the claim follows.

Lemma 2.2: \cite{10}
If $R$ is a right (left) weakly $\pi$-regular ring, then $J(R)$ is a nil ideal.

Proposition 2.3:
Let $R$ be a weakly $\pi$-regular ring and for every $a \in R$, $r(a^n) \subset r(a)$. Then, $a R = a Ra^n R$, for some positive integer $n$.

Proof:
Assume that $R$ is a weakly $\pi$-regular, then for every $a \in R$, there exist $b, c \in R$ such that $a^n = a^n ba^n c$, for some positive integer $n$. Therefore, $a^n (1 - ba^n c) = 0$. So, $1 - ba^n c \in r(a^n) \subset r(a)$. Thus, $a(1 - ba^n c) = 0$ and hence it follows that $a = a ba^n c$.

Theorem 2.4:
Let $R$ be a semi-prime ring such that each non-zero right ideal contains a non-zero ideal. If $R / r(a)$ is weakly $\pi$-regular ring, for every $a \in R$, then $R$ is weakly $\pi$-regular.
Proof:

Let $0 \neq a \in R$ such that $a^2 = 0$. Then, by the assumption that there is a non-zero ideal $I$ of $R$ with $I \subseteq aR$, we claim $\ell(a) \cap I \neq 0$. For if $Ia = 0$, then $I \subseteq \ell(a)$. If $Ia \neq 0$, then $Iaa = 0$ implies $Ia \subseteq \ell(a)$. But, since $R$ is semi-prime, then $0 \neq (\ell(a) \cap I)^2 \subseteq \ell(a)I \subseteq \ell(a)aR = 0$ is a contradiction. Consequently, $R$ is reduced. Now, assume that $R/r(a)$ is weakly $\pi$-regular, for every $a \in R$ and a positive integer $n$, $a+r(a) \in R/r(a)$, there exist $b+r(a), c+r(a) \in R/r(a)$ such that

$\sum_{k=0}^{n-1} a^k + r(a) = (\sum_{k=0}^{n-1} a^k + r(a)) (b + r(a)) (\sum_{k=0}^{n-1} a^k + r(a)) (c + r(a))$

$= a^n ba^n c + r(a)$. Then, $(a^n - a^n ba^n c) \in r(a)$. This implies that $a(a^n - a^n ba^n c) = 0$. Therefore, $(1 - ba^n c) \in r(a^{n+1}) = r(a^{n+1})$ since $R$ is reduced, then $a^n = a^n ba^n c$. Whence $R$ is weakly $\pi$-regular.

3. The Connection Between Weakly $\pi$-Regular Rings and Other Rings

A ring $R$ is said to be abelian if every idempotent element of $R$ is central. Warfield[11] introduced the class of exchange rings, where a ring $R$ is an exchange ring if the right regular module $R$ has the finite exchange property and proved that the definition is left-right symmetric. In[9] the following result is proved.

Lemma 3.1:
Every strongly $\pi$-regular ring is right weakly $\pi$-regular.

Lemma 3.2:[5]
Any abelian exchange ring is quasi duo.

Now, we investigate the relationship between the weakly $\pi$-regular and strongly $\pi$-regular in exchange rings.

Theorem 3.3:
Let $R$ be an abelian and exchange ring. Then, $R$ is a right weakly $\pi$-regular if and only if $R$ is a strongly $\pi$-regular.

Proof:

Assume that $R$ is a weakly $\pi$-regular ring. Then, for every $x \in R$, there exists a positive integer $n$ such that $x^nR = x^nRx^nR$. Since $R$ is an abelian and exchange ring, then $R$ is a right quasi duo by Lemma 3.2. Now, we claim that $x^nR + r(x^n) = R$. If not, there is a maximal right ideal $M$ of $R$ such that $x^nR + r(x^n) \subseteq M$. Then, $x^nR = x^nM$ and so $x^n = x^n y$, for some $y \in M$. Hence $x^n(1 - y) = 0$ and so $(1 - y) \in r(x^n) \subseteq M$, which is a contradiction. Therefore, $R$ is a strongly $\pi$-regular ring.

Conversely, assume that $R$ is a strongly $\pi$-regular ring, then by Lemma 3.1, $R$ is a weakly $\pi$-regular ring.

Lemma 3.4:[4]
A ring $R$ is a right quasi duo if and only if $R/J(R)$ is a reduced ring.

Furthermore, we have the following proposition.

Proposition 3.5:
Let $R$ be an abelian and exchange ring. Then, the following statements are equivalent:

1- $R$ is strongly $\pi$-regular.
2. $R$ is $\pi$-regular.
3. $R$ is a right (left) weakly $\pi$-regular.
4. $R/J(R)$ is a strongly regular ring with nil $J(R)$.

Proof:

$\Rightarrow (2) \Rightarrow (3)$. It is directly verified.

$(3) \Rightarrow (4)$. Assume that $R$ is a right (left) weakly $\pi$-regular. Since $R$ is an abelian and exchange ring, then by Theorem 3.3, $R$ is strongly $\pi$-regular and hence, $R/J(R)$ is strongly $\pi$-regular. Also, by Lemma 3.4, $R/J(R)$ is reduced and hence it is strongly regular. By Lemma 2.2, $J(R)$ is nil.

$(4) \Rightarrow (1)$. Assume that $R/J(R)$ is a strongly regular ring. It follows from (Corollary 2, 6, Theorem 3) and Theorem 3.3 that $R$ is a strongly $\pi$-regular ring.

Recall that a ring $R$ is right Kasch if every maximal right ideal is a right annihilator.

Proposition 3.6:
If a right Kasch ring $R$ is reversible, then the following statements are equivalent:

1. $R$ is strongly $\pi$-regular.
2. $R$ is right weakly $\pi$-regular.

Proof:
From Lemma 3.2, it follows that (1) implies (2).

Now, we show that, $(2) \Rightarrow (1)$. Let $M$ be any maximal right ideal of $R$. Since $R$ is a right Kasch, then $M = r(a)$, for some $a \in R$. For any $x \in M$ we have $ax=0$. Since $R$ is reversible, then $xa=0$ and $xar=0$ for all $r \in R$, so $arx=0$ and $aRx=0$. This implies that $Rx \subseteq r(a)=M$, which proves that $M$ is an ideal of $R$. Assume that $R$ is right weakly $\pi$-regular. Then, for every $b \in R$, there exists a positive integer $n$ such that $b^n R b^n R = b^n R$. We claim that $bR + r(b^n) = R$. If not, there is a maximal right ideal $N$ of $R$ such that $bR + r(b^n) \subseteq N$. Then, $b^n R = b^n N$, and so $b^n = b^n c$, for some $c \in M$. Hence $b^n (1-c) = 0$ and so $(1-c) \in r(b^n) \subseteq N$, which is a contradiction. Whence $R$ is strongly $\pi$-regular.

Recall that a ring $R$ is $N$-ring (also called a 2-primal ring) if it is a prime radical that coincides with the set of all nilpotent elements of $R$. Recall also that a ring $R$ is called bounded index of nilpotency if there exists a positive integer $n$ such that $a^n = 0$, for all nilpotent elements $a$ of $R$.

Lemma 3.7:[12]
Let $R$ be a right quasi duo ring of bounded index of nilpotency with $J(R)$ nil. Then, $R$ is $N$-ring.

Lemma 3.8:[4]
Let $R$ be a right quasi duo ring. If every prime of $R$ is maximal then $R$ is strongly $\pi$-regular.

Proposition 3.9:
Let $R$ be a reversible, right Kasch ring of bounded index of nilpotency. Then, the following statement is equivalent:

1. $R$ is strongly $\pi$-regular.
2. $R$ is $\pi$-regular.
3- $R/J(R)$ is $\pi$-regular and $J(R)$ is nil.
4- $R$ is right (left) weakly $\pi$-regular.
5- $R/J(R)$ is right weakly $\pi$-regular.
6- $R/P(R)$ is right weakly $\pi$-regular.
7- Every prime ideal of $R$ is maximal.

Proof:

By Proposition 3.6, a ring $R$ is right (or left) weakly $\pi$-regular if and only if $R$ is strongly $\pi$-regular, and if and only if $R$ is $\pi$-regular when $R$ is reversible, right Kasch ring, we obtain (1) $\iff$ (4) and (1) $\iff$ (2). Also by Proposition 3.6, (3) $\implies$ (5).

(2) $\implies$ (3), (4) $\implies$ (5) and (4) $\implies$ (6). These are obvious.

(5) $\implies$ (1). Assume $R/P(R)$ is right weakly $\pi$-regular, then by Proposition 3.6, $R/P(R)$ is strongly $\pi$-regular. Thus, $R$ is strongly $\pi$-regular (Theorem 2.1).[2]

(6) $\implies$ (7). Assume $R/J(R)$ is right weakly $\pi$-regular and $J(R)$ is nil. Since $J(R)$ is nil, then by Lemma 3.7, $R$ is $N$-ring, $J(R) = P(R)$. By (4), every prime ideal of $R$ is maximal.

(7) $\implies$ (1). Assume that (7) holds. Let $P$ be a prime ideal of $R$. Then, $P$ is maximal. From Lemma 3.8, $R$ is strongly $\pi$-regular.

Definition 3.10:[13]

Let $I$ be an arbitrary ideal of the ring $R$. We say that the idempotents of $R/I$ can be lifted into $R$ in case every idempotent element of $R/I$ is of the form $e + I$, where $e$ is an idempotent element of $R$.

Theorem 3.11:

Let $R$ be an abelian locally finite ring. Then, the following statements are equivalent:

a- $R$ is strongly $\pi$-regular.
b- $R$ is $\pi$-regular.
c- $R$ is right (left) weakly $\pi$-regular.
d- $J(R)$ is right (left) weakly $\pi$-regular and $J(R)$ is nil.

Proof:

(a) $\implies$ (b) $\implies$ (c) $\implies$ (d). It is obvious.

(b) $\implies$ (a). Assume that $R/J(R)$ is a right weakly $\pi$-regular ring, then by Theorem 3.3, $R/J(R)$ is strongly $\pi$-regular. Also, by Lemma 3.11, $R/J(R)$ is reduced and so it is strongly regular. Therefore, for each $x \in R$, there exists $y \in R$ such that $(x - xyx) \in J(R)$. Denote $x + J(R)$ by $x^-$. By hypothesis, every idempotent in $R/J(R)$ can be lifted to an idempotent of $R$. Since $J(R)$ is a nil ideal, then there exists $e \in R$ such that $e = x^2 = e \in R$ and we obtain $x^- = x = x^- 2y = x^- y = x = ex^-$. But $x-ex$ is nilpotent, and there exists a positive integer $n$ such that $(x-ex)^n = 0$. Thus, $x^n \in eR$ because $e$ is central (abelian), i.e., $x^n R \subseteq eR$, $e^2 = x^2y = x^- y = x^- 2y = \ldots = x^n - y^n$. Then, it follows that $(e-x^n) \in J(R)$ and $(e-x^n)^m = 0$ because $J(R)$ is a nil ideal, for some positive integer $m$. Now, we have $e \in x^n R$, i.e., $eR \subseteq x^n R$ and, consequently, $x^n R = eR$. Since $R$ is an abelian, then $R$ is strongly $\pi$-regular.
Now we investigate the relationship between the weakly $\pi$-regular ring and the maximality of prime ideals in locally finite rings.

**Lemma 3.12:**
Let $R$ be a locally finite abelian ring. If $R$ is of bounded index of nilpotency, then $R$ is $N$-ring with $P(R) = J(R) = N(R)$.

**Lemma 3.13:**
Let $R$ be $N$-ring. Then, $R/P(R)$ is a right weakly $\pi$-regular if and only if every prime ideal is a maximal.

**Proposition 3.14:**
Let $R$ be a locally finite abelian ring. If $R$ is of bounded index of nilpotency, then the following statements are equivalent:
1. $R/J(R)$ is right weakly $\pi$-regular and $J(R)$ is nil.
2. Every prime ideal of $R$ is maximal.

**Proof:**
(1) $\Rightarrow$ (2). Assume (1). By Lemma 3.13, $P(R) = N(R) = J(R)$. By Lemma 3.12, every prime ideal of $R$ is maximal.
(2) $\Rightarrow$ (1). Assume (2). By Lemma 3.14, $R/P(R)$ is right weakly $\pi$-regular and so $R/J(R)$ is right weakly $\pi$-regular. By Lemma 3.13, $P(R) = J(R) = N(R)$, $J(R)$ is nil.

**Lemma 3.15:**
Let $R$ be an abelian right quasi duo ring. Then, the following statements are equivalent:
1. $R$ is a strongly $\pi$-regular ring.
2. $R$ is $\pi$-regular ring.
3. $R$ is a right (left) weakly $\pi$-regular ring.
4. $R/J(R)$ is a Von Neumann regular ring with nil $J(R)$.
5. $R/J(R)$ is a strongly regular ring with nil $J(R)$.

The following proposition extends Lemma 3.15.

**Proposition 3.16:**
Let $R$ be a locally finite abelian ring. If $R$ is of bounded index of nilpotency, then the following conditions are equivalent:
1. $R$ is strongly $\pi$-regular.
2. $R$ is $\pi$-regular.
3. $R$ is right weakly $\pi$-regular.
4. $R/J(R)$ is a Von Neumann regular ring with nil $J(R)$.
5. $R/J(R)$ is a strongly regular ring with nil $J(R)$.
6. $R/J(R)$ is right weakly $\pi$-regular with nil $J(R)$.
7. $R/P(R)$ is right weakly $\pi$-regular.
8. Every prime ideal of $R$ is maximal.

**Proof:**
It is an immediate consequence of Lemma 3.15 and Proposition 3.14.

According to [15], a one-sided ideal of a ring $R$ is said to have the insertion of factors-principal (or simply IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. Hence, we shall call a ring $R$ an IFP ring if the zero ideal of $R$ has the IFP.

**Lemma 3.17:**
If $R$ is an IFP ring, then $P(R) = N(R)$.

**Theorem 3.18:**
Let $R$ be an IFP ring. Then, the following statements are equivalent:
a- $R/J(R)$ is right weakly $\pi$-regular and $J(R)$ is nil.

b- Every prime ideal of $R$ is maximal.

Proof:

(a) $\Rightarrow$ (b). Assume (a). Since $R$ is an IFP ring, then by Lemma 3.17, $P(R) = N(R)$. Also $J(R)$ is nil, $J(R) = P(R)$. By Lemma 3.13, every prime ideal of $R$ is maximal.

(b) $\Rightarrow$ (a). Assume (b). By Lemma 3.13, $R/ P(R)$ is right weakly $\pi$-regular and so $R/J(R)$ is right weakly $\pi$-regular. Now, let $a \in J(R)$. Consider $a^* \in R^* = R/ P(R)$. Since $R^*$ is right weakly $\pi$-regular, there exists a positive integer $n$ such that $a^* \ R^* = a^* \ R^* a^* \ R^*$. So, $a^* \ n = a^* \ b^* \in R^* a^* \ R^* \subseteq J^*(R)$, where $J^*(R) = J(R)/ P(R)$. Then, $a^* \ (J^* - b^*) = 0^*$ and so $a^* \in P(R)$. Since $R$ is an IFP, $a \in P(R)$, and hence $J(R)$ is nil.

References


